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Path integral of the hydrogen atom, the Jacobi's principle of least action and one-dimensional quantum gravity

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Abstract

A path integral evaluation of the Green's function for the hydrogen atom initiated by Duru and Kleinert is studied by recognizing it as a special case of the general treatment of the separable Hamiltonian of Liouville-type. The basic dynamical principle involved is identified as the Jacobi's principle of least action for given energy which is reparametrization invariant, and thus the appearance of a gauge freedom is naturally understood. The separation of variables in operator formalism corresponds to a choice of gauge in path integral, and the Green's function is shown to be gauge independent if the operator ordering is properly taken into account. Unlike the conventional Feynman path integral,which deals with a space-time picture of particle motion, the path integral on the basis of the Jacobi's principle sums over orbits in space. We illustrate these properties by evaluating an exact path integral of the Green's function for the hydrogen atom in parabolic coordinates, and thus avoiding the use of the Kustaanheimo-Stiefel transformation. In the present formulation , the Hamiltonian for Stark effect is converted to the one for anharmonic oscillators with an unstable quartic coupling. We also study the hydrogen atom path integral from a view point of one-dimensional quantum gravity coupled to matter fields representing the electron coordinates. A simple BRST analysis of the problem with an evaluation of Weyl anomaly is presented .

1 Introduction

The Hamiltonian of the hydrogen atom provides one of those few examples which can be solved exactly, and as such various interesting alternative methods to solve it have been proposed in the past. Among those methods, one may count an elegant momentum space analysis by Schwinger[1]who exploited the O(4) symmetry hidden in the hydrogen atom Hamiltonian. The path integral treatment of the problem is also interesting not only for a methodological interest but also for a pedagogical purpose. A comprehensive study of a semi-classical approximation of the path integral for the Green's function at a given energy has been performed by Gutzwiller[2]; in particular, he found an exact Green's function for negative energy in the polar coordinates of momentum space by a semi-classical approximation. It is also known that a path integral of the s-wave propagator (or evolution operator) is obtained by summing a perturbative series [3]. In 1979, Duru and Kleinert[4] showed an elegant path integral method to evaluate the Green's function for the hydrogen atom exactly. Two basic ingredients in their method are the use of a re-scaled time variable and the so called Kustaanheimo-Stiefel transformation[5]which reveals the O(4) symmetry explicitly in the coordinate space. Many of the clarifying works of this approach have been published [6] - [15]. The main issue in these works is a physical meaning of the "re-scaled time variable". We here study this issue from a completely different view point on the basis of the Jacobi's principle of least action by using a general gauge theoretical technique.

We first recognize the procedure in Ref.[4] as a special case of the general treatment of classically separable Hamiltonian of Liouville-type. The basic dynamical principle involved is then identified as the Jacobi's principle of least action for given energy. The fundamental feature of the conventional Feynman path integral, which is associated with the Hamilton's principle of stationary action, is that it deals with a *space-time* picture of particle motion. On the other hand, the path integral on the basis of the Jacobi's principle of least action is basically static and analogous to geometrical optics. A space-time picture is thus lost, and one deals with a sum over orbits in *space* instead of space-time. Another characteristics of the Jacobi's principle of least action is that it is reparametrization invariant, and the appearance of a gauge freedom to fix an arbitrary parameter,

which describes orbits for fixed energy, is clearly seen. The general technique of gauge theory is thus applicable to the evaluation of path integral, and a suitable choice of gauge simplifies the problem such as the hydrogen atom. In particular, the Green's function is shown to be gauge independent.

The use of the Kustaanheimo-Stiefel transformation is rather technical and it is not essential in solving the problem exactly. We in fact show a simple trick in parabolic coordinates which solves the hydrogen atom exactly. This trick was used before in a different context by Ravndal and Toyoda[16]. We also note that the path integral in holomorphic variables (or the coherent-state path integral) is convenient to evaluate the path integral of the hydrogen atom.

As an application of the present approach, the Hamiltonian of Stark effect is shown to be reduced to anharmonic oscillators with an unstable quartic coupling, for which a resummation technique of a perturbative series is well-known. We also study the hydrogen atom path integral from a view point of one-dimensional quantum gravity, and a simple BRST analysis of the problem with an evaluation of the Weyl anomaly is presented. This provides an alternative way to see the gauge independence of Green's functions.

2 Separable Hamiltonian of Liouville-type

We here explain the basic procedure to treat a general separable Hamiltonian of Liouville-type. We start with a separable Hamiltonian

$$H = \frac{1}{V_1(q_1) + V_2(q_2)} \{ \frac{1}{2m}(p_1^2 + p_2^2) + U_1(q_1) + U_2(q_2) \} \tag{2.1}$$

where the variables change over $\infty > q_1, q_2 > -\infty$. A general Hamiltonian of Liouville-type is given by

$$H = \frac{1}{V_1(Q_1) + V_2(Q_2)} \{ \frac{1}{2mW_1(Q_1)} P_1^2 + \frac{1}{2mW_2(Q_2)} P_2^2 + U_1(Q_1) + U_2(Q_2) \} \tag{2.2}$$

but after a canonical transformation

$$\begin{aligned} \frac{1}{\sqrt{W_1(Q_1)}} P_1 = p_1 \quad &, \quad \int_0^{Q_1} \sqrt{W_1(Q)} dQ = q_1 \\ \frac{1}{\sqrt{W_2(Q_2)}} P_2 = p_2 \quad &, \quad \int_0^{Q_2} \sqrt{W_2(Q)} dQ = q_2 \end{aligned} \tag{2.3}$$

and a suitable redefinition of V and U, we can derive a Hamiltonian of the form in (2.1).

We may then solve the Schroedinger problem

$$E\psi = \frac{1}{V_1(q_1)+V_2(q_2)}\{\frac{1}{2m}(\hat{p}_1^2+\hat{p}_2^2)+U_1(q_1)+U_2(q_2)\}\psi \tag{2.4}$$

where

$$\hat{p}_l = -i\hbar\frac{\partial}{\partial q_l} \tag{2.5}$$

for $l = 1, 2$,and the volume element dV, which renders the Hamiltonian H in (2.4) hermitian, is given by

$$dV = (V_1(q_1)+V_2(q_2))dq_1dq_2 \tag{2.6}$$

The classical Hamiltonian (2.1) does not completely specify the operator ordering in (2.4), and the simplest ordering is adopted here. A precise operator ordering needs to be fixed depending on each explicit example; a concrete example shall be given for the hydrogen atom later.

One may rewrite the above Schroedinger equation (2.4) as

$$\hat{H}_T\psi = 0 \tag{2.7}$$

with a *total Hamiltonian* defined by a specific gauge condition,

$$\hat{H}_T = \frac{1}{2m}(\hat{p}_1^2+\hat{p}_2^2)+U_1(q_1)+U_2(q_2)-E(V_1(q_1)+V_2(q_2)) \tag{2.8}$$

The meaning of a total Hamiltonian is clarified later. A general procedure to deal with a completely separated operator $\hat{H}_T$ is to consider an evolution operator for a parameter τ defined by

$$\begin{aligned}
\langle q_{1b},q_{2b}|e^{-i\hat{H}_T\tau/\hbar}|q_{1a},q_{2a}\rangle &= \langle q_{1b}|\exp[-(i/\hbar)(\frac{1}{2m}\hat{p}_1^2+U_1(q_1)-EV_1(q_1))\tau]|q_{1a}\rangle \\
&\times \langle q_{2b}|\exp[-(i/\hbar)(\frac{1}{2m}\hat{p}_2^2+U_2(q_2)-EV_2(q_2))\tau]|q_{2a}\rangle \\
&= \int \mathcal{D}q_1\mathcal{D}p_1 e^{(i/\hbar)\int_0^\tau\{p_1\dot{q}_1-(\frac{1}{2m}p_1^2+U_1(q_1)-EV_1(q_1))\}d\tau} \\
&\times \int \mathcal{D}q_2\mathcal{D}p_2 e^{(i/\hbar)\int_0^\tau\{p_2\dot{q}_2-(\frac{1}{2m}p_2^2+U_2(q_2)-EV_2(q_2))\}d\tau}
\end{aligned} \tag{2.9}$$

The parameter τ is arbitrary , and by integrating over τ from 0 to ∞ one obtains a physically meaningful quantity

$$\langle q_{1b},q_{2b}|\frac{\hbar}{\hat{H}_T}|q_{1a},q_{2a}\rangle_{semi-classical}$$

$$
\begin{aligned}
&= i\int_0^\infty d\tau \frac{1}{\sqrt{2\pi i\hbar(\partial q_1(\tau)/\partial p_1(0))_{q_{1a}}}} \frac{1}{\sqrt{2\pi i\hbar(\partial q_2(\tau)/\partial p_2(0))_{q_{2a}}}} \\
&\times \exp\{(i/\hbar)S_{cl}(q_{1b},q_{1a},\tau)+(i/\hbar)S_{cl}(q_{2b},q_{2a},\tau)\} \qquad (2.10)
\end{aligned}
$$

where we wrote the result of a semi-classical approximation for the path integral[17][18] [19], though in certain cases one may be able to perform an exact path integral in (2.9). The pre-factor in (2.10) is written in terms of classical paths , for example,

$$
q_{1cl}(\tau) = q_1(\tau; q_{1a}, p_1(0)) \qquad (2.11)
$$

Namely, the classical paths dictated by the total Hamiltonian $\hat{H}_T$ are expressed as functions of the initial positions and momenta. On the other hand, the classical action S_{cl} is expressed as a function of the initial position, final position and elapsed "time" τ by eliminating $p_1(0)$ dependence;for example,

$$
S_{cl}(q_{1b},q_{1a},\tau) = \int_0^\tau \{p_1\dot{q}_1 - (\frac{1}{2m}p_1^2 + U_1(q_1) - EV_1(q_1))\}_{cl} d\tau \qquad (2.12)
$$

with $q_1(\tau) = q_{1b}$. If one solves the Hamilton-Jacobi equation in the form

$$
S(q_{1b},q_{1a};\tau) = -A\tau + S(q_{1b},q_{1a};A) \qquad (2.13)
$$

one treats A as a dynamical variable and regards the above equation as a Legendre transformation defined by

$$
\begin{aligned}
\frac{\partial S(q_{1b},q_{1a};\tau)}{\partial \tau} &= -A \\
\frac{\partial S(q_{1b},q_{1a};A)}{\partial A} &= \tau \qquad (2.14)
\end{aligned}
$$

The variable A is then eliminated. This may be regarded as a classical analogue of uncertainty relation; if one specifies τ, the conjugate variable A becomes implicit. It is known that the semi-classical approximation (2.10) is exact for a quadratic system such as a simple harmonic oscillator[18].

We next note the relation for the quantity defined in the left-hand side of (2.9)

$$
\begin{aligned}
&\langle q_{1b},q_{2b}|\frac{1}{\hat{H}_T}|q_{1a},q_{2a}\rangle \\
&= \langle q_{1b},q_{2b}|\frac{1}{(\frac{1}{\hat{V}_1(q_1)+\hat{V}_2(q_2)})\hat{H}_T}|q_{1a},q_{2a}\rangle \frac{1}{V_1(q_{1a})+V_2(q_{2a})} \\
&= \langle q_{1b},q_{2b}|\frac{1}{\hat{H}-E}|q_{1a},q_{2a}\rangle \frac{1}{V_1(q_{1a})+V_2(q_{2a})} \qquad (2.15) \\
&= \frac{1}{H(q_{1b},\frac{\hbar}{i}\frac{\partial}{\partial q_{1b}},..)-E}\{\frac{1}{\sqrt{V_1(q_{1b})+V_2(q_{2b})}}\langle q_{1b},q_{2b}|q_{1a},q_{2a}\rangle \frac{1}{\sqrt{V_1(q_{1a})+V_2(q_{2a})}}\}
\end{aligned}
$$

by recalling $(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}$. The state vectors in these relations are defined for the volume element $dq_1 dq_2$ as

$$\int dq_1 dq_2 |q_1, q_2\rangle\langle q_1, q_2| = 1 \tag{2.16}$$

Note that the definition of the δ-function in $\langle q_1', q_2'|q_1, q_2\rangle = \delta(q_1' - q_1)\delta(q_2' - q_2)$ depends on the choice of the volume elememt in (2.16) and thus on the choice of H_T. The last expression in (2.15) is thus correctly defined for the original Hamiltonian H and the original state ψ in (2.4) with the volume element dV in (2.6), since we have the completeness relation from (2.16)

$$\int |q_1, q_2\rangle \frac{dV}{V_1(q_1) + V_2(q_2)} \langle q_1, q_2| = 1 \tag{2.17}$$

The left-hand side of (2.15) thus defines the correct Green's function for the original operator $(\hat{H} - E)^{-1}$ by noting the symmetry in q_a and q_b. One can then define the conventional evolution operator by

$$\begin{aligned} &\langle q_{1b}, q_{2b}|e^{-i\hat{H}(t_b - t_a)/\hbar}|q_{1a}, q_{2a}\rangle_{conv} \\ &= \frac{1}{2\pi i\hbar}\int_{-\infty}^{\infty} dE e^{-iE(t_b - t_a)/\hbar}\langle q_{1b}, q_{2b}|\frac{\hbar}{\hat{H} - i\epsilon - E}|q_{1a}, q_{2a}\rangle \frac{1}{V_1(q_{1a}) + V_2(q_{2a})} \end{aligned} \tag{2.18}$$

where ϵ is an infinitesimal positive number. The total Hamiltonian changes for a different choice of gauge condition in the Jacobi's principle of least action to be explained below. Consequently, the volume element, which renders H_T hermitian, generally depends on the choice of gauge. In this case, one has the relation

$$\begin{aligned} \langle q_{1b}, q_{2b}|q_{1a}, q_{2a}\rangle_{conv} &= \langle q_{1b}, q_{2b}|q_{1a}, q_{2a}\rangle \frac{1}{V_1(q_{1a}) + V_2(q_{2a})} \\ &= \frac{1}{\sqrt{V_1(q_{1b}) + V_2(q_{2b})}}\langle q_{1b}, q_{2b}|q_{1a}, q_{2a}\rangle \frac{1}{\sqrt{V_1(q_{1a}) + V_2(q_{2a})}} \end{aligned}$$

The meaning of the total Hamiltonian H_T (2.8) becomes transparent if one starts with the Jacobi's principle of least action for a given E

$$\begin{aligned} S &= \int_0^\tau d\tau L = \int_0^\tau d\tau \sqrt{2m[E(V_1(q_1) + V_2(q_2)) - (U_1(q_1) + U_2(q_2))](\dot{q}_1^2 + \dot{q}_2^2)} \\ &= \int \sqrt{2m[E(V_1(q_1) + V_2(q_2)) - (U_1(q_1) + U_2(q_2))][(dq_1)^2 + (dq_2)^2]} \end{aligned} \tag{2.19}$$

which is reparametrization invariant. One then defines the momenta conjugate to coordinates

$$p_l = \frac{\partial L}{\partial \dot{q}_l} = \sqrt{2m[E(V_1(q_1) + V_2(q_2)) - (U_1(q_1) + U_2(q_2))]} \times \frac{\dot{q}_l}{\sqrt{(\dot{q}_1^2 + \dot{q}_2^2)}} \tag{2.20}$$

and obtains a vanishing Hamiltonian, which is a result of reparametrization invariance, and a first class constraint ϕ as the generator of reparametrization gauge symmetry,

$$\begin{aligned} H &= p_l\dot{q}_l - L = 0 \\ \phi(q_l, p_l) &= \frac{1}{V_1(q_1)+V_2(q_2)}\{\frac{1}{2m}(p_1^2+p_2^2)+U_1(q_1)+U_2(q_2)\} - E \simeq 0 \end{aligned} \tag{2.21}$$

Following Dirac[20], one may then define a total Hamiltonian

$$\begin{aligned} H_T &= H + \alpha(q_l, p_l)\phi(q_l, p_l) \\ &= \alpha(q_l, p_l)\phi(q_l, p_l) \simeq 0 \end{aligned} \tag{2.22}$$

where an arbitrary function $\alpha(q_l, p_l)$ specifies a choice of gauge or a choice of the arbitrary parameter τ in (2.19), which parametrizes the orbit for a given E. The quantum theory is defined by(up to an operator ordering)

$$i\hbar\frac{\partial}{\partial\tau}\psi = \hat{H}_T\psi \tag{2.23}$$

with a physical state condition

$$\hat{\alpha}(q_l, p_l)\hat{\phi}(q_l, p_l)\psi_{phy} = 0 \tag{2.24}$$

A choice of the specific gauge $\alpha(q_l, p_l) = V_1(q_1) + V_2(q_2)$ gives rise to (2.7) and the choice $\alpha(q_l, p_l) = 1$ gives the conventional static Schroedinger equation (2.4), since ψ appearing in these equations are physical states.

The basic dynamical principle involved is thus identified as the Jacobi's principle of least action, which is analogous to geometrical optics, and the formula of an evolution operator (2.9) dictated by (2.23) provides a basis for the path integral approach to a general separable Hamiltonian of Liouville-type. The path integral in (2.9) deals with a sum over orbits in space instead of space-time, and the notion of re-scaled time does not explicitly appear in the present approach; the evolution operator (2.9) essentially generates a gauge transformation.

3 Hydrogen Atom

3.1, Analysis in Parabolic Coordinates

We analyze the hydrogen atom by starting with the Hamiltonian written in terms of parabolic coordinates

$$H(\xi,\eta,\varphi)=\frac{1}{2m(\xi+\eta)}(\xi p_\xi^2+\eta p_\eta^2)+\frac{1}{8m\xi\eta}p_\varphi^2-\frac{e^2}{\xi+\eta} \tag{3.1}$$

where the parabolic coordinates are introduced via the cylindrical coordinates (ρ,φ,z) by

$$\begin{aligned}\xi&=\frac{1}{2}(r-z)\\ \eta&=\frac{1}{2}(r+z)\\ r&=\sqrt{\rho^2+z^2}\end{aligned} \tag{3.2}$$

and φ stands for the azimuthal angle around the z axis. We further perform a canonical transformation which simplifies the kinetic term in H as

$$\begin{aligned}\xi &= \frac{1}{4}u^2 \qquad ,0\le u<\infty\\ \sqrt{\xi}p_\xi &= p_u\\ \eta &= \frac{1}{4}v^2 \qquad ,0\le v<\infty\\ \sqrt{\eta}p_\eta &= p_v\end{aligned} \tag{3.3}$$

and the Hamiltonian becomes

$$H=\frac{1}{2m}(\frac{4}{u^2+v^2})[p_u^2+\frac{1}{u^2}p_\varphi^2+p_v^2+\frac{1}{v^2}p_\varphi^2]-\frac{4}{u^2+v^2}e^2 \tag{3.4}$$

where $r=\xi+\eta=(u^2+v^2)/4$. This Hamiltonian is not yet a separable one of Liouville-type.

One may solve the Schroedinger equation

$$\hat{H}\psi=E\psi \tag{3.5}$$

or equivalently

$$\hat{H}_T\psi=0 \tag{3.6}$$

with

$$\hat{H}_T=\frac{1}{2m}[\hat{p}_u^2+\frac{1}{u^2}\hat{p}_\varphi^2+\hat{p}_v^2+\frac{1}{v^2}\hat{p}_\varphi^2]-e^2+\frac{m\omega^2}{2}(u^2+v^2) \tag{3.7}$$

where ω is defined by

$$\frac{1}{2}m\omega^2=-\frac{1}{4}E \tag{3.8}$$

We consider the case $E < 0$ for the moment. $\hat{H}_T$ stands for the total Hamiltonian defined by a specific gauge condition; a general definition of $\hat{H}_T$ will be given later in (3.47).

Eq.(3.6) may be rewritten in an equivalent form as

$$\begin{aligned} \hat{\tilde{H}}_T \psi &= 0 \\ (\hat{p}_\varphi - \hat{p}_{\varphi'})\psi &= 0 \end{aligned} \tag{3.9}$$

We here introduced auxiliary variables $(\hat{p}_{\varphi'}, \varphi')$ as

$$\begin{aligned} \hat{\tilde{H}}_T &= \frac{1}{2m}[\hat{p}_u^2 + \frac{1}{u^2}\hat{p}_\varphi^2 + \hat{p}_v^2 + \frac{1}{v^2}\hat{p}_{\varphi'}] + \frac{m\omega^2}{2}(u^2+v^2) - e^2 \\ &= \frac{1}{2m}\vec{p}_u^{\,2} + \frac{m\omega^2}{2}\vec{u}^2 + \frac{1}{2m}\vec{p}_v^{\,2} + \frac{m\omega^2}{2}\vec{v}^2 - e^2 \end{aligned} \tag{3.10}$$

and we defined

$$\begin{aligned} \vec{u} &= (u_1, u_2) = (u\cos\varphi, u\sin\varphi) \\ \vec{p}_u^{\,2} &= \hat{p}_u^2 + \frac{1}{u^2}\hat{p}_\varphi^2 \\ \vec{v} &= (v_1, v_2) = (v\cos\varphi', v\sin\varphi') \\ \vec{p}_v^{\,2} &= \hat{p}_v^2 + \frac{1}{v^2}\hat{p}_{\varphi'}^2 \end{aligned} \tag{3.11}$$

The subsidiary condition in (3.9) replaces the use of the Kustaanheimo-Stiefel transformation, and at the same time it renders a Hamiltonian of Liouville-type. This introduction of auxiliary variables (3.11) has been discussed by Ravndal and Toyoda[16]. The use of the subsidiary condition (3.9) in place of the Kustaanheimo-Stiefel transformation may be useful for a pedagogical purpose.

A general procedure to deal with a completely separated operator such as $\hat{\tilde{H}}_T$ in (3.10) is to consider an evolution operator for a parameter τ defined by

$$\begin{aligned} &\langle \vec{u}_b, \vec{v}_b | e^{-i\hat{\tilde{H}}_T\tau/\hbar} | \vec{u}_a, \vec{v}_a \rangle \\ &= e^{ie^2\tau} \langle \vec{u}_b | exp[-(i/\hbar)(\frac{1}{2m}\vec{p}_u^{\,2} + \frac{m\omega^2}{2}\vec{u}^2)\tau] | \vec{u}_a \rangle \\ &\times \langle \vec{v}_b | exp[-(i/\hbar)(\frac{1}{2m}\vec{p}_v^{\,2} + \frac{m\omega^2}{2}\vec{v}^2)\tau] | \vec{v}_a \rangle \\ &= e^{ie^2\tau}(\frac{m\omega}{2\pi i\hbar\sin\omega\tau})^{4/2} \\ &\times exp\{\frac{im\omega}{2\hbar\sin\omega\tau}[(\vec{u}_b^{\,2} + \vec{v}_b^{\,2} + \vec{u}_a^{\,2} + \vec{v}_a^{\,2})\cos\omega\tau - 2\vec{u}_b\vec{u}_a - 2\vec{v}_b\vec{v}_a]\} \end{aligned} \tag{3.12}$$

where we used the exact result for a simple harmonic oscillator[18]

$$\begin{aligned}&\langle q_b|exp[-(i/\hbar)(\frac{1}{2m}\hat{p}^2+\frac{m\omega^2}{2}\hat{q}^2)\tau]|q_a\rangle\\&=(\frac{m\omega}{2\pi i\hbar\sin\omega\tau})^{1/2}exp\{\frac{im\omega}{2\hbar\sin\omega\tau}[(q_b^2+q_a^2)\cos\omega\tau-2q_bq_a]\}\end{aligned}\tag{3.13}$$

which can be established either in path integral or in operator formalism(see (2.10)). Also, the harmonic oscillator is symmetric with respect to coordinate or momentum space representation [2].

A crucial observation here is that $\hat{p}_\varphi$ and $\hat{p}_{\varphi'}$ are preserved during the evolution dictated by the operator $\hat{\tilde{H}}_T$ in (3.10), since $[\hat{p}_\varphi,\hat{\tilde{H}}_T]=[\hat{p}_{\varphi'},\hat{\tilde{H}}_T]=0$. It is then sufficient to impose the constraint (3.9) only on the initial state , for example. Starting with a general state belonging to the eigenvalues $\hat{p}_\varphi=m$ and $\hat{p}_{\varphi'}=m'$

$$e^{im\varphi}e^{im'\varphi'}\tag{3.14}$$

we can use the following trick

$$\int_0^{2\pi}\frac{d\theta}{2\pi}e^{im(\varphi+\theta)}e^{im'(\varphi'-\theta)}=\delta_{m,m'}e^{im(\varphi+\varphi')}\tag{3.15}$$

to project out the state satisfying $\hat{p}_\varphi=\hat{p}_{\varphi'}$, and $\varphi+\varphi'$ is regarded as the actual azimuthal angle.

We thus obtain

$$\begin{aligned}&\langle u_b,v_b,(\varphi+\varphi')_b|e^{-i\hat{H}_T\tau/\hbar}|u_a,v_a,(\varphi+\varphi')_a\rangle\\&=e^{ie^2\tau}(\frac{m\omega}{2\pi i\hbar\sin\omega\tau})^2\int_0^{2\pi}\frac{d\theta}{2\pi}exp\{\frac{im\omega}{2\hbar\sin\omega\tau}[(\vec{u}_b^2+\vec{v}_b^2+\vec{u}_a^2+\vec{v}_a^2)\cos\omega\tau-2\vec{u}_b\vec{u}_a-2\vec{v}_b\vec{v}_a]\}\\&=e^{ie^2\tau}(\frac{m\omega}{2\pi i\hbar\sin\omega\tau})^2\times\\&\int_0^{2\pi}\frac{d\theta}{2\pi}exp\{\frac{im\omega}{2\hbar\sin\omega\tau}[4(\xi_a+\xi_b+\eta_a+\eta_b)\cos\omega\tau-4\sqrt{2}(r_ar_b+\vec{x}_a\vec{x}_b)^{1/2}\cos(\theta+\gamma)]\}\\&=e^{ie^2\tau}(\frac{m\omega}{2\pi i\hbar\sin\omega\tau})^2exp\{\frac{2im\omega}{\hbar\sin\omega\tau}(r_a+r_b)\cos\omega\tau\}I_0(\frac{2\sqrt{2}im\omega}{\hbar\sin\omega\tau}(r_ar_b+\vec{x}_a\vec{x}_b)^{1/2})\end{aligned}\tag{3.16}$$

In this evaluation we start with the relation

$$\vec{u}_b\vec{u}_a+\vec{v}_b\vec{v}_a=u_bu_a\cos\Delta\varphi+v_bv_a\cos\Delta\varphi'\tag{3.17}$$

with $\Delta\varphi = \varphi_b - \varphi_a, \Delta\varphi' = \varphi'_b - \varphi'_a$, and

$$
\begin{aligned}
& u_b u_a \cos(\Delta\varphi + \theta) + v_b v_a \cos(\Delta\varphi' - \theta) \\
& = (u_b u_a \cos\Delta\varphi + v_b v_a \cos\Delta\varphi')\cos\theta \\
& +(-u_b u_a \sin\Delta\varphi + v_b v_a \sin\Delta\varphi')\sin\theta \\
& = 4\sqrt{\xi_b\xi_a + \eta_b\eta_a + 2\sqrt{\xi_b\xi_a\eta_b\eta_a}\cos(\Delta\varphi + \Delta\varphi')}\cos(\theta+\gamma) \\
& = 2\sqrt{2}\sqrt{r_a r_b + z_a z_b + \rho_a\rho_b\cos(\Delta\varphi + \Delta\varphi')}\cos(\theta+\gamma) \\
& = 2\sqrt{2}\sqrt{r_a r_b + \vec{x}_a\vec{x}_b}\cos(\theta+\gamma)
\end{aligned}
\tag{3.18}
$$

where we used the definition of variables in (3.3), and γ is a number independent of θ. We also defined a modified Bessel function

$$
I_0(\frac{2\sqrt{2}im\omega}{\hbar\sin\omega\tau}(r_a r_b + \vec{x}_a\vec{x}_b)^{1/2}) = \int_0^{2\pi}\frac{d\theta}{2\pi}exp\{\frac{2\sqrt{2}im\omega(r_a r_b + \vec{x}_a\vec{x}_b)^{1/2}}{\hbar\sin\omega\tau}\cos\theta\}
\tag{3.19}
$$

The parameter τ is arbitrary, and we eliminate τ to obtain a physically meaningful quantity by

$$
\begin{aligned}
& i\int_0^\infty d\tau\langle u_b, v_b, (\varphi+\varphi')_b|e^{-i\hat{H}_T\tau/\hbar}|u_a, v_a, (\varphi+\varphi')_a\rangle \\
& = \langle u_b, v_b, (\varphi+\varphi')_b|\frac{\hbar}{\hat{H}_T}|u_a, v_a, (\varphi+\varphi')_a\rangle \\
& = i\int_0^\infty d\tau e^{ie^2\tau}(\frac{m\omega}{2\pi i\hbar\sin\omega\tau})^2 exp\{\frac{2im\omega}{\hbar\sin\omega\tau}(r_a+r_b)\cos\omega\tau\} \\
& \times I_0(\frac{2\sqrt{2}im\omega}{\hbar\sin\omega\tau}(r_a r_b + \vec{x}_a\vec{x}_b)^{1/2}) \\
& = \frac{m\omega}{2\pi^2\hbar^2}\int_0^1 d\lambda\lambda^{-\nu}\frac{1}{(1-\lambda)^2}exp[\frac{-2m\omega}{\hbar}(r_a+r_b)(\frac{1+\lambda}{1-\lambda})]I_0(\frac{4\sqrt{2}m\omega}{\hbar}\frac{\lambda^{1/2}}{1-\lambda}(r_a r_b + \vec{x}_a\vec{x}_b)^{1/2})
\end{aligned}
\tag{3.20}
$$

where we rotated τ by 90 degrees , $\tau \to -i\tau$, and defined

$$
\begin{aligned}
\lambda &= e^{-2\omega\tau} \\
\nu &= e^2/2\omega
\end{aligned}
\tag{3.21}
$$

We next show that (3.20) gives an exact Green's function for the hydrogen atom by noting the sequence

$$
\langle u_b, v_b, \varphi_b|\frac{\hbar}{\hat{H}_T}|u_a, v_a, \varphi_a\rangle
$$

$$
\begin{aligned}
&= \langle \xi_b, \eta_b, \varphi_b | \frac{\hbar}{(\frac{1}{\hat{\xi}+\hat{\eta}})\hat{H}_T(\xi,\eta,\varphi)} | \xi_a, \eta_a, \varphi_a \rangle (\frac{1}{\xi_a+\eta_a}) \\
&= \langle \xi_b, \eta_b, \varphi_b | \frac{\hbar}{\hat{H}(\xi,\eta,\varphi) - E} | \xi_a, \eta_a, \varphi_a \rangle (\frac{1}{\xi_a+\eta_a}) \\
&= \frac{\hbar}{\hat{H}(\xi_b,\eta_b,\varphi_b) - E} (\frac{1}{\sqrt{\xi_b+\eta_b}} \langle \xi_b, \eta_b, \varphi_b | \xi_a, \eta_a, \varphi_a \rangle \frac{1}{\sqrt{\xi_a+\eta_a}}) \\
&= \frac{1}{4\pi} \langle \vec{x}_b | \frac{\hbar}{\hat{\vec{p}}^2/2m - e^2/r - E} | \vec{x}_a \rangle
\end{aligned} \tag{3.22}
$$

where we used φ in place of $\varphi + \varphi'$ and the relation $(\hat{A}\hat{B})^{-1} = \hat{B}^{-1}\hat{A}^{-1}$.

The volume element changes in this transition from $\hat{H}_T$ to $\hat{H}$ as

$$
\begin{aligned}
dV_0 &= 2\pi uv du dv d\varphi \\
\rightarrow dV &= (\xi+\eta) dV_0 = 4\pi \times 2(\xi+\eta) d\xi d\eta d\varphi \\
&= 4\pi \times r^2 dr d\cos\theta d\varphi
\end{aligned} \tag{3.23}
$$

The bra- and ket- vectors in (3.22) are normalized in the combination

$$
\begin{aligned}
\int dV_0 |u,v,\varphi\rangle\langle u,v,\varphi| &= 1 \\
\int dV |\xi,\eta,\varphi\rangle \frac{1}{\xi+\eta} \langle \xi,\eta,\varphi| &= 1 \\
\int d^3x |\vec{x}\rangle\langle\vec{x}| &= 1
\end{aligned} \tag{3.24}
$$

and the extra factor of 4π in $dV = 4\pi r^2 dr d\cos\theta d\varphi$ requires the appearance of the factor of $1/4\pi$ in the last expression in (3.22). The appearance of 2π in dV_0 is an artifact of the variable φ' in (3.11). This normalization condition of bra- and ket- vectors together with a symmetry in $\vec{x}_a$ and $\vec{x}_b$ justify the identification (3.22). A more explicit and concrete analysis of eqs.(3.22)$\sim$ (3.24) will be given in connection with the Jacobi's principle later.

As for the operator ordering, the momentum operator changes in (3.22) as

$$
\begin{aligned}
\hat{p}_u^2 + \hat{p}_v^2 &= (\frac{\hbar}{i})^2 [\frac{1}{u}\partial_u u \partial_u + \frac{1}{v}\partial_v v \partial_v] \\
&= (\frac{\hbar}{i})^2 [\partial_\xi \xi \partial_\xi + \partial_\eta \eta \partial_\eta] \\
&= \hat{p}_\xi \xi \hat{p}_\xi + \hat{p}_\eta \eta \hat{p}_\eta
\end{aligned} \tag{3.25}
$$

and

$$
(\frac{1}{\xi+\eta})(\hat{p}_\xi \xi \hat{p}_\xi + \hat{p}_\eta \eta \hat{p}_\eta) + \frac{1}{4\xi\eta}\hat{p}_\varphi^2 = \hat{\vec{p}}^2 \tag{3.26}
$$

where the right-hand side is written in cartesian coordinates. We note that dV_0 and dV in (3.23) respectively render $\hat{H}_T$ and $\hat{H}(\xi,\eta,\varphi)$ hermitian.

Combining (3.20),(3.22) and (3.26), we have thus established the exact Green's function including the operator ordering

$$\begin{aligned}\langle\vec{x}_b|\frac{\hbar}{\hat{\vec{p}}^2/2m-e^2/r-E}|\vec{x}_a\rangle &= \frac{2m^2\omega}{\pi\hbar^2}\int_0^1 d\lambda\lambda^{-\nu}\frac{1}{(1-\lambda)^2}exp[\frac{-2m\omega}{\hbar}(r_a+r_b)(\frac{1+\lambda}{1-\lambda})] \\ &\quad\times I_0(\frac{4\sqrt{2}m\omega}{\hbar}\frac{\lambda^{1/2}}{1-\lambda}(r_ar_b+\vec{x}_a\vec{x}_b)^{1/2}) \qquad (3.27)\end{aligned}$$

It is known that this formula, which was first derived by Duru and Kleinert[4], is a Fourier transform of Schwinger's momentum space representation[1]. The continuation to the scattering problem with $E>0$ is performed by the replacement

$$\omega\to(-i)\omega,\quad \nu\to i\nu \qquad (3.28)$$

in the above formula.

One can understand the spectrum of the hydrogen atom by looking at $\tilde{H}_T$ in (3.10). This problem, which has been analyzed by Ravndal and Toyoda[16], is briefly summarized here in connection with the Jacobi's principle of least action and the Stark effect to be discussed later. If one defines the oscillator variables

$$\begin{aligned}a_k &= \frac{1}{\sqrt{2}}[\sqrt{\frac{m\omega}{\hbar}}u_k+\frac{i}{\sqrt{m\omega\hbar}}\hat{p}_{u_k}], \\ \tilde{a}_k &= \frac{1}{\sqrt{2}}[\sqrt{\frac{m\omega}{\hbar}}v_k+\frac{i}{\sqrt{m\omega\hbar}}\hat{p}_{v_k}],\quad k=1,2 \qquad (3.29)\end{aligned}$$

one obtains

$$\begin{aligned}\hat{\tilde{H}}_T &= \hbar\omega[\sum_{k=1}^{2}(a_k^\dagger a_k+\tilde{a}_k^\dagger\tilde{a}_k)+2]-e^2 \\ \hat{p}_\varphi &= i\hbar[a_1^\dagger a_2-a_2^\dagger a_1] \\ \hat{p}_{\varphi'} &= i\hbar[\tilde{a}_1^\dagger\tilde{a}_2-\tilde{a}_2^\dagger\tilde{a}_1] \qquad (3.30)\end{aligned}$$

After a unitary transformation

$$\begin{aligned}a_1 &= \frac{1}{\sqrt{2}}(b_1-ib_2) \\ a_2 &= \frac{1}{\sqrt{2}}(-ib_1+b_2) \qquad (3.31)\end{aligned}$$

and a similar transformation of $\tilde{a}_1$ and $\tilde{a}_2$, one obtains

$$\begin{aligned}
\hat{\tilde{H}}_T &= \hbar\omega[\sum_{k=1}^{2}(b_k^\dagger b_k + \tilde{b}_k^\dagger \tilde{b}_k) + 2] - e^2 \\
\hat{p}_\varphi &= \hbar[b_1^\dagger b_1 - b_2^\dagger b_2] \\
\hat{p}_{\varphi'} &= \hbar[\tilde{b}_1^\dagger \tilde{b}_1 - \tilde{b}_2^\dagger \tilde{b}_2]
\end{aligned} \tag{3.32}$$

By defining the number operators

$$\begin{aligned}
n_k &= b_k^\dagger b_k, \\
\tilde{n}_k &= \tilde{b}_k^\dagger \tilde{b}_k, \quad k = 1, 2
\end{aligned} \tag{3.33}$$

the total Hamiltonian is given by

$$\begin{aligned}
\hat{\tilde{H}}_T &= \hbar\omega[n_1 + n_2 + \tilde{n}_1 + \tilde{n}_2 + 2] - e^2 \\
&= \hbar\omega[2n_1 - \hat{p}_\varphi/\hbar + 2\tilde{n}_2 + \hat{p}_{\varphi'}/\hbar + 2] - e^2 \\
&= 2\hbar\omega[n_1 + \tilde{n}_2 + 1] - e^2
\end{aligned} \tag{3.34}$$

by noting the physical state condition $\hat{p}_\varphi = \hat{p}_{\varphi'}$.

We thus define the principal quantum number (or its operator) n by

$$n = n_1 + \tilde{n}_2 + 1 = 1, 2, 3, ... \tag{3.35}$$

and the physical state condition

$$(2n\hbar\omega - e^2)\psi_{phys} = 0 \tag{3.36}$$

gives rise to the Bohr spectrum

$$E = -\frac{mc^2}{2}(\frac{e^2}{\hbar c})^2 \frac{1}{n^2}, \quad n = 1, 2, 3, ... \tag{3.37}$$

by noting the definition of ω in (3.8).

As for the degeneracy of states with a fixed n, we have n combinations of $(n_1, \tilde{n}_2)$ given by

$$\begin{aligned}
\tilde{n}_2 &= n - (n_1 + 1), \\
n_1 &= 0, 1, ..., n - 1
\end{aligned} \tag{3.38}$$

For each fixed $(n_1, \tilde{n}_2)$, we have the constraints

$$\begin{aligned} n_2 &= n_1 - \hat{p}_\varphi/\hbar \geq 0 \\ \tilde{n}_1 &= \tilde{n}_2 + \hat{p}_{\varphi'}/\hbar = n - (n_1+1) + \hat{p}_{\varphi'}/\hbar \geq 0 \end{aligned} \tag{3.39}$$

which gives

$$n_1 \geq \hat{p}_\varphi/\hbar = \hat{p}_{\varphi'}/\hbar \geq -n + (n_1+1) \tag{3.40}$$

Namely, we have

$$n_1 - (-n + n_1 + 1) + 1 = n \tag{3.41}$$

possible values of $\hat{p}_\varphi$. We thus have

$$n \times n = n^2 \tag{3.42}$$

degeneracy of states with a fixed principal quantum number n, as required. This somewhat unorthodox classification of states may be useful in the analysis of the Stark effect to be commented on later.

Incidentally, the formula (3.32) suggests that one can define the Green's function by

$$i\int_0^\infty d\tau \int_0^{2\pi} d\theta \langle f|exp\{-\frac{i}{\hbar}\{\hbar\omega[\sum_{k=1}^{2}(b_k^\dagger b_k + \tilde{b}_k^\dagger \tilde{b}_k) + 2] - e^2\}\tau - i(b_1^\dagger b_1 - b_2^\dagger b_2 - \tilde{b}_1^\dagger \tilde{b}_1 + \tilde{b}_2^\dagger \tilde{b}_2)\theta\}|i\rangle \tag{3.43}$$

for a direct path integral in (3.16), instead of using the trick noted in (3.15). The θ integral imposes the constraint $\hat{p}_\varphi = \hat{p}_{\varphi'}$. The path integral of an evolution operator in terms of oscillator variables is known as holomorphic (or coherent state) path integral[21]. It is interesting that (3.43) resembles a closed string propagation on a cylindrical world-sheet.

3.2, Jacobi's Principle of Least Action

The meaning of the total Hamiltonian in (3.7) becomes transparent , if one starts with a Nambu-Goto-type Lagrangian (the Jacobi's principle of least action for a given E) which is reparametrization invariant,

$$\begin{aligned} S &= \int_0^\tau L d\tau = \int_0^\tau d\tau \sqrt{2m(E - V(r))(\frac{d\vec{x}}{d\tau})^2} \\ &= \int \sqrt{2m(E - V(r))(d\vec{x})^2} \end{aligned} \tag{3.44}$$

The momenta conjugate to coordinates are then defined by

$$\vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = \sqrt{2m(E - V(r))}(\frac{d\vec{x}}{d\tau})/\sqrt{(\frac{d\vec{x}}{d\tau})^2} \tag{3.45}$$

and one obtains a vanishing Hamiltonian as a result of reparametrization invariance and a first-class constraint ϕ, which is the generator of reparametrization gauge symmetry,

$$\begin{aligned} H &= \vec{p}\dot{\vec{x}} - L = 0 \\ \phi(\vec{x},\vec{p}) &= \frac{\vec{p}^2}{2m} + V(r) - E \simeq 0 \end{aligned} \tag{3.46}$$

Following Dirac[20], the total Hamiltonian is defined by

$$\begin{aligned} H_T &= H + \alpha(\vec{x},\vec{p})\phi(\vec{x},\vec{p}) \\ &= \alpha(\vec{x},\vec{p})\phi(\vec{x},\vec{p}) \simeq 0 \end{aligned} \tag{3.47}$$

and the function $\alpha(\vec{x},\vec{p})$ specifies a choice of gauge and fixes the arbitrary parameter τ in (3.44), which parametrizes the orbit for a given E. A change of the parameter τ to $\tau - \delta\beta(\tau,\vec{x},\vec{p})$ is generated by $\delta\beta(\tau,\vec{x},\vec{p})\alpha(\vec{x},\vec{p})\phi(\vec{x},\vec{p}) = \delta\beta(\tau,\vec{x},\vec{p})H_T$, for example,

$$\begin{aligned} \delta\vec{x}(\tau) &= \vec{x}'(\tau) - \vec{x}(\tau) \\ &= \vec{x}(\tau+\delta\beta) - \vec{x}(\tau) \\ &= \{\vec{x}, \delta\beta H_T\}_{PB} \\ &= \delta\beta(\tau,\vec{x},\vec{p})\frac{d}{d\tau}\vec{x}(\tau) \end{aligned} \tag{3.48}$$

in terms of the Poisson bracket, since $\vec{x}'(\tau-\delta\beta) = \vec{x}(\tau)$.

Quantization is performed by

$$i\hbar\frac{\partial}{\partial\tau}\psi = \hat{H}_T\psi \tag{3.49}$$

with a physical state condition

$$\hat{\alpha}(\vec{x},\vec{p})\hat{\phi}(\vec{x},\vec{p})\psi_{phy} = 0 \tag{3.50}$$

A specific choice of the gauge $\alpha(\vec{x},\vec{p}) = r = \xi+\eta$ leads to the Hamiltonian $\hat{H}_T$ in (3.7) and the choice $\alpha(\vec{x},\vec{p}) = 1$ gives the original static Schroedinger equation(3.5), since the states ψ in (3.7) and (3.5) are physical states. Eq.(3.49) gives rise to the evolution operator in (3.12), but the parameter τ need not be interpreted as a re-scaled time. In fact, the evolution operator (3.12), which essentially generates a gauge transformation, deals with a sum over orbits in space instead of space-time.

We now explain the relations (3.22)$\sim$(3.24) in a more concrete manner. We start with eq.(3.22) for a generic negative E

$$\begin{aligned} G(E;\xi_b,\eta_b,\varphi_b;\xi_a,\eta_a,\varphi_a) &= \langle \xi_b,\eta_b,\varphi_b| \frac{\hbar}{\hat{H}_T(E;\xi,\eta,\varphi)} |\xi_a,\eta_a,\varphi_a\rangle \\ &\equiv \sum_n \phi_n(E;\xi_b,\eta_b,\varphi_b) \frac{\hbar}{\lambda_n(E)} \phi_n^\star(E;\xi_a,\eta_a,\varphi_a) \end{aligned} \tag{3.51}$$

with

$$\begin{aligned} \hat{H}_T(E;\xi,\eta,\varphi)\phi_n(E;\xi,\eta,\varphi) &= \lambda_n(E)\phi_n(E;\xi,\eta,\varphi) \\ \int \phi_n^\star(E;\xi,\eta,\varphi)\phi_l(E;\xi,\eta,\varphi)dV_0 &= \delta_{n,l} \\ \lambda_n(E) = 2n\hbar\sqrt{-\frac{E}{2m}} - e^2 & \\ dV_0 = 4\pi \times 2d\xi d\eta d\varphi & \end{aligned} \tag{3.52}$$

where we used the result in (3.34) and also the variables (ξ,η,φ) instead of (u,v,φ) for notational simplicity. The summation over n in (3.51) is formal including the n^2 degeneracy. Note that the complete orthonormal states $\{\phi_n\}$ in (3.52) are all *unphysical* off- shell states. In path integral, the summation in (3.51) is exactly evaluated in (3.20).

We next rewrite $G(E;\xi_b,\eta_b,\varphi_b;\xi_a,\eta_a,\varphi_a)$ in terms of physical on-shell states by writing an unsubtracted dispersion relation (i.e., paying attention only to the pole structure in E) as

$$G(E;\xi_b,\eta_b,\varphi_b;\xi_a,\eta_a,\varphi_a) = \sum_n \phi_n(E_n;\xi_b,\eta_b,\varphi_b) \frac{\hbar}{(E_n-E)(-\frac{\partial\lambda_n(E_n)}{\partial E_n})} \phi_n^\star(E_n;\xi_a,\eta_a,\varphi_a) \tag{3.53}$$

by noting

$$\lambda_n(E) = \lambda_n(E_n) + (E-E_n)\frac{\partial\lambda_n(E_n)}{\partial E_n} = (E-E_n)(\frac{-e^2}{2E_n}) \tag{3.54}$$

for $E \approx E_n$.

When one defines

$$\psi_n(E_n;\xi,\eta,\varphi) = \frac{1}{\sqrt{-\frac{\partial\lambda_n(E_n)}{\partial E_n}}} \phi_n(E_n;\xi,\eta,\varphi) \tag{3.55}$$

one can show the orthonormality relations

$$\begin{aligned} \int \psi_n^\star(E_n;\xi,\eta,\varphi)\psi_l(E_l;\xi,\eta,\varphi)(\xi+\eta)dV_0 &= \int \psi_n^\star(E_n;\xi,\eta,\varphi)\psi_l(E_l;\xi,\eta,\varphi)dV \\ &= \delta_{n,l} \end{aligned} \tag{3.56}$$

with $dV = (\xi + \eta)dV_0$. First of all, from the physical state condition

$$\begin{aligned} & \hat{H}_T(E_n;\xi,\eta,\varphi)\phi_n(E_n;\xi,\eta,\varphi) \qquad (3.57) \\ = \; & \{\frac{1}{2m}(\hat{p}_\xi \xi \hat{p}_\xi + \hat{p}_\eta \eta \hat{p}_\eta) + \frac{1}{8m}(\frac{1}{\xi}+\frac{1}{\eta})\hat{p}_\varphi^2 - E_n(\xi+\eta) - e^2\}\phi_n(E_n;\xi,\eta,\varphi) = 0 \end{aligned}$$

one can establish the orthogonality relation

$$(E_n - E_l)\int \phi_n^\star(E_n;\xi,\eta,\varphi)\phi_l(E_l;\xi,\eta,\varphi)(\xi+\eta)dV_0 = 0 \qquad (3.58)$$

for $n \neq l$. Also from the relation (3.57) and the fact that the "eigenvalue" e^2 is equally distributed for the kinetic and potential terms for harmonic oscillators(in terms of u and v variables) , we have

$$-E_n \int \phi_n^\star(E_n;\xi,\eta,\varphi)\phi_n(E_n;\xi,\eta,\varphi)(\xi+\eta)dV_0 = \frac{e^2}{2} \qquad (3.59)$$

namely

$$\frac{1}{-\frac{\partial\lambda_n(E_n)}{\partial E_n}}\int \phi_n^\star(E_n;\xi,\eta,\varphi)\phi_n(E_n;\xi,\eta,\varphi)(\xi+\eta)dV_0 = 1 \qquad (3.60)$$

by noting (3.54). This proves (3.56).

From (3.53), we finally arrive at the expression

$$\begin{aligned} G(E;\xi_b,\eta_b,\varphi_b;\xi_a,\eta_a,\varphi_a) &= \sum_n \psi_n(E_n;\xi_b,\eta_b,\varphi_b)\frac{\hbar}{E_n - E}\psi_n^\star(E_n;\xi_a,\eta_a,\varphi_a) \\ &= \langle \xi_b,\eta_b,\varphi_b|\frac{\hbar}{\hat{H}(\xi,\eta,\varphi) - E}|\xi_a,\eta_a,\varphi_a\rangle \\ &= \frac{1}{4\pi}\langle \vec{x}_b|\frac{\hbar}{\hat{\vec{p}}^2/(2m) - e^2/r - E}|\vec{x}_a\rangle \qquad (3.61) \end{aligned}$$

which establishes the gauge independence of the Green's function for negative E. See also Ref.(14). Although we here used the same notation for the state $|\xi,\eta,\varphi\rangle$ in (3.51) and (3.61), the meaning of these states are quite different. This difference is explicitly exhibited in (3.22) $\sim$ (3.24). It is important to realize that the exact path integral is performed for the off-shell states in (3.51). As for the case of positive energy, one can define the Green's function by analytic continuation in the starting expression in (3.51) and in the final expression in (3.61).

More generally, one can establish the gauge independence of the Green's function for an arbitrary choice of gauge condition $\alpha(\xi,\eta,\varphi) = f(\xi,\eta,\varphi)$ by noting

$$\begin{aligned} \hat{H}_T(E;\xi,\eta,\varphi)\phi_n(E;\xi,\eta,\varphi) &= [\hat{f}(\xi,\eta,\varphi)\hat{H}(\xi,\eta,\varphi) - \hat{f}(\xi,\eta,\varphi)E]\phi_n(E;\xi,\eta,\varphi) \\ &= \lambda_n(E)\phi_n(E;\xi,\eta,\varphi) \qquad (3.62) \end{aligned}$$

Namely

$$\begin{aligned}\int \phi_n^\star(E;\xi,\eta,\varphi)\hat{H}_T(E;\xi,\eta,\varphi)\phi_n(E;\xi,\eta,\varphi)dV_f &= \lambda_n(E)\int \phi_n^\star(E;\xi,\eta,\varphi)\phi_n(E;\xi,\eta,\varphi)dV_f \\ &= \lambda_n(E) \qquad (3.63)\end{aligned}$$

with

$$dV_f = \frac{1}{f(\xi,\eta,\varphi)}dV \qquad (3.64)$$

which renders $\hat{H}_T(E;\xi,\eta,\varphi)$ defined by the gauge $\alpha = f(\xi,\eta,\varphi)$ hermitian. From (3.62) and (3.63), one derives

$$\begin{aligned}\frac{\partial\lambda_n(E)}{\partial E} &= \int \phi_n^\star(E;\xi,\eta,\varphi)\frac{\partial}{\partial E}\hat{H}_T(E;\xi,\eta,\varphi)\phi_n(E;\xi,\eta,\varphi)dV_f \\ &= -\int \phi_n^\star(E;\xi,\eta,\varphi)\phi_n(E;\xi,\eta,\varphi)f(\xi,\eta,\varphi)dV_f \\ &= -\int \phi_n^\star(E;\xi,\eta,\varphi)\phi_n(E;\xi,\eta,\varphi)dV \qquad (3.65)\end{aligned}$$

If one uses this relation for $E = E_n$ in (3.53), one arrive at the expression (3.61) starting with an arbitrary gauge condition $\alpha = f(\xi,\eta,\varphi)$.

4 A related topic : Stark effect

As an interesting implication of the present treatment of the separable Hamiltonian of Liouville-type, we comment on the Stark effect (the hydrogen atom inside a constant external electric field $\mathcal{E}$) described by a Hamiltonian

$$\begin{aligned}H &= \frac{1}{2m}\vec{p}^2 - \frac{e^2}{r} - e\mathcal{E}z \\ &= \frac{1}{2m(\xi+\eta)}(\xi p_\xi^2 + \eta p_\eta^2) + \frac{1}{8m\xi\eta}p_\varphi{}^2 - \frac{e^2}{\xi+\eta} - e\mathcal{E}(\eta-\xi) \qquad (4.1)\end{aligned}$$

We thus analyze the total Hamiltonian defined by

$$\hat{H}_T = \frac{1}{2m}\vec{p}_u^2 + \frac{m\omega^2}{2}\vec{u}^2 + \frac{1}{2m}\vec{p}_v^2 + \frac{m\omega^2}{2}\vec{v}^2 - e^2 - \frac{1}{4}g(\vec{v}^4 - \vec{u}^4) \qquad (4.2)$$

with a constraint

$$\hat{p}_\varphi - \hat{p}'_\varphi = 0 \qquad (4.3)$$

for the coodinates defined in (3.10). The quartic coupling constant g is given by

$$g = \frac{1}{4}e\mathcal{E} \qquad (4.4)$$

We can thus analyze the Stark effect on the basis of

$$
\begin{aligned}
& \langle \vec{u}_b, \vec{v}_b | e^{-i\hat{H}_T \tau/\hbar} | \vec{u}_a, \vec{v}_a \rangle \\
= \; & e^{ie^2\tau} \langle \vec{u}_b | e^{-(i/\hbar)(\frac{1}{2m}\hat{\vec{p}}_u^{\,2} + \frac{m\omega^2}{2}\vec{u}^2 + \frac{1}{4}g\vec{u}^4)\tau} | \vec{u}_a \rangle \\
& \times \langle \vec{v}_b | e^{-(i/\hbar)(\frac{1}{2m}\hat{\vec{p}}_v^{\,2} + \frac{m\omega^2}{2}\vec{v}^2 - \frac{1}{4}g\vec{v}^4)\tau} | \vec{v}_a \rangle \\
= \; & e^{ie^2\tau} \int \mathcal{D}\vec{p}_u \mathcal{D}\vec{u} \exp\{(i/\hbar) \int_0^\tau [\vec{p}_u \dot{\vec{u}} - (\frac{1}{2m}\vec{p}_u^{\,2} + \frac{m\omega^2}{2}\vec{u}^2 + \frac{1}{4}g\vec{u}^4)] d\tau\} \\
& \times \int \mathcal{D}\vec{p}_v \mathcal{D}\vec{v} \exp\{(i/\hbar) \int_0^\tau [\vec{p}_v \dot{\vec{v}} - (\frac{1}{2m}\vec{p}_v^{\,2} + \frac{m\omega^2}{2}\vec{v}^2 - \frac{1}{4}g\vec{v}^4)] d\tau\}
\end{aligned}
\tag{4.5}
$$

This problem in the conventional formulation has been recently analyzed by K. Hiraizumi, Y. Ohshima and H. Suzuki[22] as an application of the resummation technique of perturbation series for quantum tunneling [23], which was established for a system of anharmonic oscillators[24]. It is interesting that this problem is in fact *identical* to the (two-dimensional) anharmonic oscillator with an unstable quartic coupling in the present formulation.

5 One-dimensional quantum gravity

A way alternative to (3.44) to see the physical meaning of the parameter τ is to study the one-dimensional quantum gravity coupled to matter variables $\vec{x}$ defined by

$$
\int \frac{\mathcal{D}\vec{x}\mathcal{D}h}{gauge\ volume} exp\{i \int_0^\tau Lh d\tau\} \tag{5.1}
$$

with

$$
L = \frac{m}{2h^2}(\frac{d\vec{x}}{d\tau})^2 - V(r) + E \tag{5.2}
$$

where h stands for the einbein, a one-dimensional analogue of vierbein h_μ^a, and $h = \sqrt{g}$ in one-dimension. In this Section, we set $\hbar = 1$. If one uses the solution of the equation of motion for h defined by the Lagrangian $\mathcal{L} = Lh$, the action in (5.1) is reduced to the one appearing in the Jacobi's principle of least action (3.44). See Ref.[25] for a related problem in the context of a relativistic particle. The canonical Liouville measure is not reparametrization invariant in general[26], and it needs to be proved. This problem is analogous to the Polyakov-type path integral in string theory[27] - [29]. We here show that the naive canonical Liouville measure is gauge invariant up to a renormalization of

the cosmological term (or energy eigenvalue E). This renormalization is however universal for any algebraic gauge fixing of the form

$$h(\tau) = f(\vec{x}(\tau)) \tag{5.3}$$

for a gauge fixing function $f(\vec{x}(\tau))$. The analysis presented below, which is formal but is known to work in string theory where no simple discretization is known, may be useful to understand certain formal aspects such as gauge invariance in the point-particle path integral.

We here analyze the BRST invariant path integral for (5.1) by using the Faddeev-Popov procedure, namely, we replace the naive measure in (5.1) by

$$\begin{aligned}
&\int \mathcal{D}(\sqrt{h}\vec{x})\mathcal{D}\sqrt{h}\mathcal{D}(h^{3/2}c)\mathcal{D}\bar{c}\mathcal{D}B \\
&\times exp\{i\int_0^\tau Lhd\tau + i\int_0^\tau [B(\sqrt{h}-\sqrt{f}) - i\frac{1}{2}\bar{c}\sqrt{h}\partial_\tau c - i\bar{c}c\partial_\tau(\sqrt{h}-\sqrt{f})]d\tau\} \\
&\equiv \int d\mu \exp[i\int_0^\tau \mathcal{L}_{eff}d\tau]
\end{aligned} \tag{5.4}$$

The BRST transformation is defined as a translation in the Grassmann parameter , $\theta \to \theta + \lambda$, in the superfield notation(note that $\theta^2 = \lambda^2 = \theta\lambda + \lambda\theta = \theta c(\tau) + c(\tau)\theta = 0$)

$$\begin{aligned}
\vec{x}(\tau,\theta) &= \vec{x}(\tau) + i\theta c(\tau)\partial_\tau \vec{x}(\tau) \\
\sqrt{h(\tau,\theta)} &= \sqrt{h(\tau)} + i\theta[c(\tau)\partial_\tau + \frac{1}{2}(\partial_\tau c(\tau))]\sqrt{h(\tau)} \\
\sqrt{h}\vec{x}(\tau,\theta) &= \sqrt{h}\vec{x}(\tau) + i\theta[c(\tau)\partial_\tau + \frac{1}{2}(\partial_\tau c(\tau))]\sqrt{h}\vec{x}(\tau) \\
\sqrt{f(\vec{x}(\tau,\theta))} &= \sqrt{f(\vec{x}(\tau))} + i\theta c(\tau)\partial_\tau\sqrt{f(\vec{x}(\tau))} \\
c(\tau,\theta) &= c(\tau) + i\theta c(\tau)\partial_\tau c(\tau) \\
\bar{c}(\tau,\theta) &= \bar{c}(\tau) + \theta B(\tau)
\end{aligned} \tag{5.5}$$

For example, the BRST transformation is given by

$$\begin{aligned}
\delta\sqrt{h(\tau)} &= i\lambda[c(\tau)\partial_\tau + \frac{1}{2}(\partial_\tau c(\tau))]\sqrt{h(\tau)} \\
\delta(\sqrt{h}\vec{x}(\tau)) &= i\lambda[c(\tau)\partial_\tau + \frac{1}{2}(\partial_\tau c(\tau))]\sqrt{h}\vec{x}(\tau) \\
\delta(h^{3/2}dc(\tau)) &= i\lambda[c(\tau)\partial_\tau + \frac{1}{2}(\partial_\tau c(\tau))](h^{3/2}dc(\tau))
\end{aligned} \tag{5.6}$$

In the last relation for the ghost variable $c(\tau)$, we consider the differential $dc(\tau)$: one may write $h^{3/2}dc = d(h^{3/2}c)$ for a fixed metric $h(\tau)$, which is the case required to study

the BRST invariance of the path integral measure in (5.4). Note that all the variables in (5.6) have the same BRST transformation law which is anomaly free[26]; the combination $\mathcal{D}\bar{c}\mathcal{D}B$ is also manifestly BRST invariant. These properties in turn ensure the BRST invariance of the path integral measure in (5.4). One can also confirm that the action in (5.4) is also BRST invariant.

By using the BRST invariance of the action and the measure in (5.4), one can prove that the path integral (5.4) is independent of the choice of $f(\vec{x}(\tau))$ as follows: Under an infinitesimal change of $f(\vec{x}(\tau))$, the path integral (5.4) changes as

$$\begin{aligned} & -i\int_0^\tau d\tau\langle[B(\tau)\delta\sqrt{f(\tau)}-i\bar{c}(\tau)c(\tau)\partial_\tau\delta\sqrt{f(\tau)}]\rangle \\ & =-i\int d\mu[B(\tau)\delta\sqrt{f(\tau)}-i\bar{c}(\tau)c(\tau)\partial_\tau\delta\sqrt{f(\tau)}]e^{i\int_0^\tau\mathcal{L}_{eff}d\tau} \end{aligned} \tag{5.7}$$

where $\langle B\delta\sqrt{f}-i\bar{c}c\partial_\tau\delta\sqrt{f}\rangle$ denotes the averaging in the path integral (5.4). Next we note the BRST identity

$$\begin{aligned} & \langle\bar{c}(\tau)\delta\sqrt{f(\vec{x}(\tau))}\rangle \\ & =\int d\mu\{\bar{c}(\tau)\delta\sqrt{f(\vec{x}(\tau))}\}e^{i\int_0^\tau\mathcal{L}_{eff}d\tau} \\ & =\int d\mu'\{\bar{c}(\tau)'\delta\sqrt{f(\vec{x}(\tau))'}\}e^{i\int_0^\tau\mathcal{L}_{eff}{}'d\tau} \\ & =\int d\mu\{\bar{c}(\tau)\delta\sqrt{f(\vec{x}(\tau))}+\lambda[B(\tau)\delta\sqrt{f(\tau)}-i\bar{c}(\tau)c(\tau)\partial_\tau\delta\sqrt{f(\tau)}]\}e^{i\int_0^\tau\mathcal{L}_{eff}d\tau} \\ & =\langle\bar{c}(\tau)\delta\sqrt{f(\vec{x}(\tau))}\rangle+\lambda\langle B(\tau)\delta\sqrt{f(\tau)}-i\bar{c}(\tau)c(\tau)\partial_\tau\delta\sqrt{f(\tau)}\rangle \end{aligned} \tag{5.8}$$

where the primed variables stand for the BRST transformed variables such as $\sqrt{h(\tau)'}=\sqrt{h(\tau)}+i\lambda[c(\tau)\partial_\tau+\frac{1}{2}(\partial_\tau c(\tau))]\sqrt{h(\tau)}$, $\delta\sqrt{f(\vec{x}(\tau))'}=\delta\sqrt{f(\vec{x}(\tau))}+i\lambda c(\tau)\partial_\tau\delta\sqrt{f(\vec{x}(\tau))}$ and $\bar{c}(\tau)'=\bar{c}(\tau)+\lambda B(\tau)$. The first equality in (5.8) holds since the path integral is independent of the naming of path integration variables, and the second equality holds because of the BRST invariance of the measure $d\mu'=d\mu$ and the effective action $\mathcal{L}_{eff}{}'=\mathcal{L}_{eff}$. (5.8) shows that $\langle B\delta\sqrt{f}-i\bar{c}c\partial_\tau\delta\sqrt{f}\rangle=0$, and thus (5.7) vanishes. Namely, the path integral (5.4) is independent of the choice of $f(\vec{x}(\tau))$, provided that one specifies the gauge invariant initial and final states.

The path integral (5.4) is rewritten as

$$\int\mathcal{D}\tilde{\vec{p}}\mathcal{D}\tilde{\vec{x}}\mathcal{D}\sqrt{h}\mathcal{D}\tilde{c}\mathcal{D}\bar{c}\mathcal{D}Bexp\{i\int_0^\tau d\tau[(\vec{p}\dot{\vec{x}}-Hh)+B(\sqrt{h}-\sqrt{f})-i\frac{1}{2}\bar{c}\sqrt{h}\partial_\tau(\frac{1}{h^{3/2}}\tilde{c})]\} \tag{5.9}$$

with

$$\begin{aligned} H &= \frac{1}{2m}\vec{p}^2 + V(r) - E \\ \tilde{\vec{p}} &= \sqrt{h}\vec{p} \\ \tilde{\vec{x}} &= \sqrt{h}\vec{x} \\ \tilde{c} &= h^{3/2}c \end{aligned} \tag{5.10}$$

In fact, after the path integral over $\tilde{\vec{p}}$, one recovers (5.4). We also set $\sqrt{h} - \sqrt{f} = 0$ after a partial integration in the ghost sector in (5.9). Note that the path integral

$$\int \mathcal{D}\tilde{\vec{p}} \exp\{i\int[-\frac{h}{2m}\vec{p}^2 + \vec{p}\dot{\vec{x}} - \frac{m}{2h}\dot{\vec{x}}^2]d\tau\} = \int \mathcal{D}\tilde{\vec{p}} \exp[i\int \frac{-1}{2m}(\tilde{\vec{p}})^2 d\tau] \tag{5.11}$$

is a constant independent of the metric $h(\tau)$: This is another way to see why the variables with weight 1/2 such as $\tilde{\vec{p}}$ and $\tilde{\vec{x}}$ are chosen for the world scalar quantities $\vec{p}$ and $\vec{x}$ as reparametrization invariant path integral variables.

In the above path integral, it is important to recognize that the singular (time-derivative) terms in the Lagrangian (5.9) are written as

$$\int \mathcal{D}\tilde{\vec{p}}\mathcal{D}\tilde{\vec{x}}\mathcal{D}\tilde{c}\mathcal{D}\bar{c} \exp\{\int_0^\tau d\tau[i\tilde{\vec{p}}\frac{1}{\sqrt{h}}\partial_\tau(\frac{1}{\sqrt{h}}\tilde{\vec{x}}) + \frac{1}{2}\bar{c}\sqrt{h}\partial_\tau(\frac{1}{h^{3/2}}\tilde{c})]\} \tag{5.12}$$

Those singular terms have a Weyl invariant structure; namely, the h-dependence can be completely removed by a suitable scale transformation of $\tilde{\vec{p}}, \tilde{\vec{x}}, \tilde{c}$ and $\bar{c}$ such as $\tilde{\vec{x}} \to \sqrt{h}\tilde{\vec{x}}$ and $\bar{c} \to (1/\sqrt{h})\bar{c}$. In this process of scaling, one obtains a Jacobian (or anomaly), which can be integrated to a Wess-Zumino term. At the same time, the path integral variables $\tilde{\vec{p}}$ and $\tilde{\vec{x}}$ are reduced to the naive ones. In the present case, one can confirm that the Jacobian (Weyl anomaly) has a form

$$M\int h\delta\beta(\tau)d\tau \tag{5.13}$$

for an infinitesimal scale transformation

$$\begin{aligned} \bar{c}(\tau) &\to e^{-\delta\beta(\tau)}\bar{c}(\tau) \\ \tilde{c}(\tau) &\to e^{3\delta\beta(\tau)}\tilde{c}(\tau) \\ \tilde{\vec{p}}(\tau) &\to e^{\delta\beta(\tau)}\tilde{\vec{p}}(\tau) \\ \tilde{\vec{x}}(\tau) &\to e^{\delta\beta(\tau)}\tilde{\vec{x}}(\tau) \end{aligned} \tag{5.14}$$

This evaluation of the Jacobian is performed for $\tilde{\vec{p}}$ and $\tilde{\vec{x}}$, for example, by[28][30]

$$\begin{aligned}
&\lim_{M\to\infty}\int\frac{dk}{2\pi}e^{-ik\tau}\exp\{(\frac{1}{\sqrt{h}}\partial_\tau\frac{1}{\sqrt{h}})^\dagger(\frac{1}{\sqrt{h}}\partial_\tau\frac{1}{\sqrt{h}})/M^2\}e^{ik\tau}\\
&=\lim_{M\to\infty}M\int\frac{dk}{2\pi}\exp\{[\frac{1}{\sqrt{h}}(\partial_\tau/M+ik)\frac{1}{\sqrt{h}}][\frac{1}{\sqrt{h}}(\partial_\tau/M+ik)\frac{1}{\sqrt{h}}]\}\\
&=\lim_{M\to\infty}M\int\frac{dk}{2\pi}\exp\{-k^2/h^2\}(1+O(\frac{1}{M^2}))\\
&=\frac{1}{2\sqrt{\pi}}Mh\quad for\ \ M\to\infty
\end{aligned}\tag{5.15}$$

which gives a term of a general structure as in (5.13). The anomaly calculation is specified by the basic operators appearing in (5.12)

$$\frac{1}{\sqrt{h}}\partial_\tau\frac{1}{\sqrt{h}}\quad ,or\ \ \sqrt{h}\partial_\tau\frac{1}{h^{3/2}}\tag{5.16}$$

both of which give the same form of anomaly proportional to h, as in (5.15). See also Ref.[28]. The overall sign of the Jacobian is specified depending on the statistics of each variable, i.e., a Grassmann variable gives an extra minus sign. Since only the most singular term survives in (5.15), one can confirm that the knowledge of the most singular terms in the Lagrangian (5.9), i.e.,the time-derivative terms in (5.12), is sufficient to calculate anomaly. If one denotes the integrated anomaly (Wess-Zumino term) by $\Gamma(h)$, we have from (5.13)

$$\Gamma(h)-\Gamma(he^{-2\delta\beta})=\int 2\delta\beta(\tau)h(\tau)\frac{\partial\Gamma}{\partial h(\tau)}d\tau=M\int h(\tau)\delta\beta(\tau)d\tau\tag{5.17}$$

since h is transformed to $he^{-2\delta\beta}$ by (5.14), and we obtain

$$\Gamma(h)=M\int h(\tau)d\tau\tag{5.18}$$

with a suitable (infinite) number M. This $\Gamma(h)$ is added to the action in (5.9) and it has a form of the cosmological (or energy) term in (5.9).

This calculation of the anomaly is quite general: In two-dimensions, the Weyl anomaly has a structure $M^2\sqrt{g}+\sqrt{g}R$,[27] - [29]. In one-dimension, the curvature term containing R does not exist, and only the cosmological term arises if one performs a reparametrization invariant calculation. This Weyl anomaly renormalizes the bare cosmological term in (5.9),

$$E+M=E_r\tag{5.19}$$

with E_r a renormalized energy parameter. But this renormalization is independent of the choice of the gauge fixing function $f(\vec{x}(\tau))$ in (5.4).

We cannot assign a physical significance to the precise value of the renormalization in (5.19), since the linear divergence in (5.15) is regularization dependent. Only the Weyl invariant structure in (5.12) for the choice of BRST invariant measure and the extraction of the metric dependence from the singular terms in (5.12) as a Wess-Zumino term have a well-defined physical meaning.

After this procedure of eliminating the h-dependence from the time-derivative terms and then the integration over B and h, one obtains a path integral

$$\int \mathcal{D}\vec{p}\mathcal{D}\vec{x}\mathcal{D}c\mathcal{D}\bar{c}exp\{i\int_0^\tau d\tau[\vec{p}\dot{\vec{x}} - f(\vec{x}(\tau))H - i\frac{1}{2}\bar{c}\partial_\tau c]\} \tag{5.20}$$

One may ignore the *decoupled* ghost sector, which is first order in ∂_τ in the present case and does not contribute to the Hamiltonian. One thus finally arrives at the phase space path integral

$$\int \mathcal{D}\vec{p}\mathcal{D}\vec{x}exp\{i\int_0^{\tau_f} d\tau[\vec{p}\dot{\vec{x}} - f(\vec{x}(\tau))H]\} \tag{5.21}$$

where the overall normalization factor, which is independent of $f(\vec{x})$ but can depend on τ_f, is fixed to be the same as for $f(\vec{x}) = 1$ in (5.21).

When one cuts the interval $[0, \tau_f]$ into meshes, the path integral measure in (5.21) is defined by

$$\begin{aligned} dV_f(\vec{x}) &= \frac{1}{f(\vec{x})}d^3x \\ dV_f(\vec{p}) &= f(\vec{x})d^3p \end{aligned} \tag{5.22}$$

for each mesh point, and one performs integral over the momentum first. Since one has one extra momentum integration relative to the coordinate one for fixed end points $\vec{x}_a(0)$ and $\vec{x}_b(\tau_f)$, the external states are also specified for the volume element $dV_f(\vec{x})$.

A way to provide gauge invariant external states is to consider a trace by integrating over $\vec{x}_a(0) = \vec{x}_b(\tau_f)$ in (5.21), and let $E_r \to E_n$. In this limit the path integral is reduced to

$$\sum \int dV_f(\vec{x}_a)\phi_n(E_n;\vec{x}_a)\phi_n^\star(E_n;\vec{x}_a) = n^2 \tag{5.23}$$

where we used the notation in (3.57), and the summation is over the n^2 degenerate states. The states $\phi_n(E_n;\vec{x})$ form a complete physical set for $\hat{H}_T(E_r = E_n)$.

A less trivial way to incorporate physical external states is to consider the Green's function

$$G(E_r;\vec{x}_b,\vec{x}_a) = i\int_0^\infty d\tau_f \int \mathcal{D}\vec{p}\mathcal{D}\vec{x} exp\{i\int_0^{\tau_f} d\tau[\vec{p}\dot{\vec{x}} - f(\vec{x}(\tau))H]\} \tag{5.24}$$

and look at the pole position at $E_r \simeq E_n$. In this case, one has (see also eq(3.53))

$$\begin{aligned}\int dV_f(\vec{x}_b)dV_f(\vec{x}_a)\phi_n^\star(E_n;\vec{x}_b)G(E_r;\vec{x}_b,\vec{x}_a)\phi_n(E_n;\vec{x}_a) &= \frac{1}{\lambda_n(E_r)} \\ &= \frac{1}{(E_r-E_n)}\frac{1}{(-\frac{d\lambda_n(E_n)}{dE_n})}\end{aligned} \tag{5.25}$$

If one recalls the definition of $\psi_n(E_n;\vec{x})$ in (3.55), one concludes(see eq.(3.61))

$$G(E_r;\vec{x}_b,\vec{x}_a) = \sum_n \psi_n(E_n;\vec{x}_b)\frac{1}{E_r-E_n}\psi_n^\star(E_n;\vec{x}_a) \tag{5.26}$$

in the sense of an unsubtracted dispersion relation. The pole positions and the residue functions in this relation are all gauge independent.

In the form of the quantity $\langle n|\frac{1}{\hat{H}_T}|n\rangle$ in (5.25), one needs to adjust the "wave function renormalization factor" $\sqrt{-\frac{d\lambda_n(E_n)}{dE_n}}$ for each external state to obtain a gauge independent physical quantity, but in the form $G(E_r;\vec{x}_b,\vec{x}_a)$ in (5.24) one need not supply the renormalization factor.

An interesting implication of the path integral approach(5.4) is that one may understand the τ-integral in (5.24) as an integral over the deformation parameter (or moduli) of an analogue of the world-line. In the present path integral, one sums over all the possible deformation of the "world-line" as well as the paths in space to obtain a physically meaningful geometrical quantity. Another interpretation of τ may be to regard it as an analogue of a proper time[31].

6 Conclusion

An attempt to solve the Green's function for the hydrogen atom exactly in the path integral[4] opened a new avenue for the path integral treatment of a general separable Hamiltonian of Liouville-type. This new view point, which has been shown to be based on the Jacobi's principle of least action , provides a more flexible framework of path integral to deal with a wider class of problems of physical interest. The Jacobi's principle

of least action , besides being reparametrization invariant, gives an attractive geometrical picture of particle orbits in a curved space deformed by the potential. On the other hand, the fundamental space-time picture of the conventional Feynman path integral, which is associated with the Hamilton's principle of stationary action, is lost. (A 4-dimensional picture is however recovered by a generalization of the Jacobi's principle for a relativistic particle, $S = -m \int d\tau \sqrt{(dx^\mu/d\tau)^2}$, and an arbitrary parameter τ is identified with a proper time in quantum theory[31]).

In the present paper, we discussed some of the basic issues related to this new approach to the path integral from a view point of general gauge theory. We have shown the gauge independence of the Green's function. A BRST analysis of the problem from a view point of one-dimensional quantum gravity is presented. We also commented on a possible application of this scheme to the Stark effect and a resummation of perturbation series.

I thank C. Bernido for calling the path integral of the hydrogen atom to my attention.

[**Note added**]

After submitting the present paper for publication, the papers quoted in Ref.[32] came to my attention. These papers deal with some related matters of the path integral treatment of the hydrogen atom.

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